Logique

Correspondance Curry-Howard

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Curry-Howard

Logique		**Programmation**
Système logique	⇔	Langage de programmation
Hypothèse	⇔	Variable
Formule	⇔	Type
Preuve	⇔	Programme

Système - Langage

Système Logique		**Langage de programmmation**
Calcul des constructions	⇔	CC (Coquand)
Logique Intuitionniste du 2e ordre	⇔	Système F (Girard)
Logique Intuitionniste du 1er ordre	⇔	Système T (Gödel)
Arithmétique Primitive Récursive	⇔	Système T_0 : Récursion primitive (Kleene)
Logique Minimale	⇔	λ-calcul simplement typé (Church)

λ-calcul

Langage de programmation élémentaire

variables : x, y, z, …

fonctions : λx.t

applications : t u

λ-calcul simplement typé

$$\frac{}{\Gamma, x : \alpha \vdash x : \alpha}$$

Ajout de la notion de type

variables : x, y, z, …

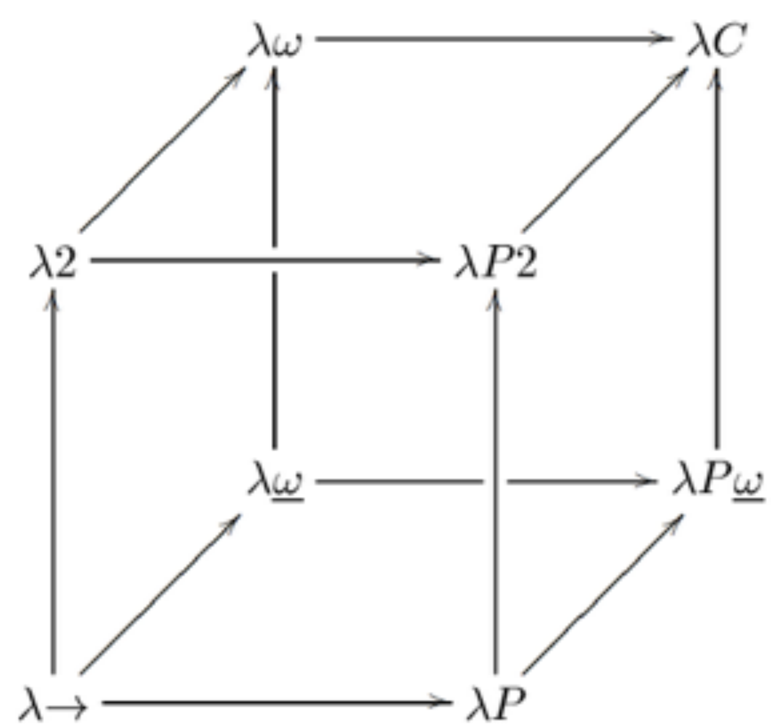
$$\frac{\Gamma, x : \alpha \vdash t : \beta}{\Gamma \vdash (\lambda x : \alpha . t) : (\alpha \to \beta)}$$

fonctions : λx.t

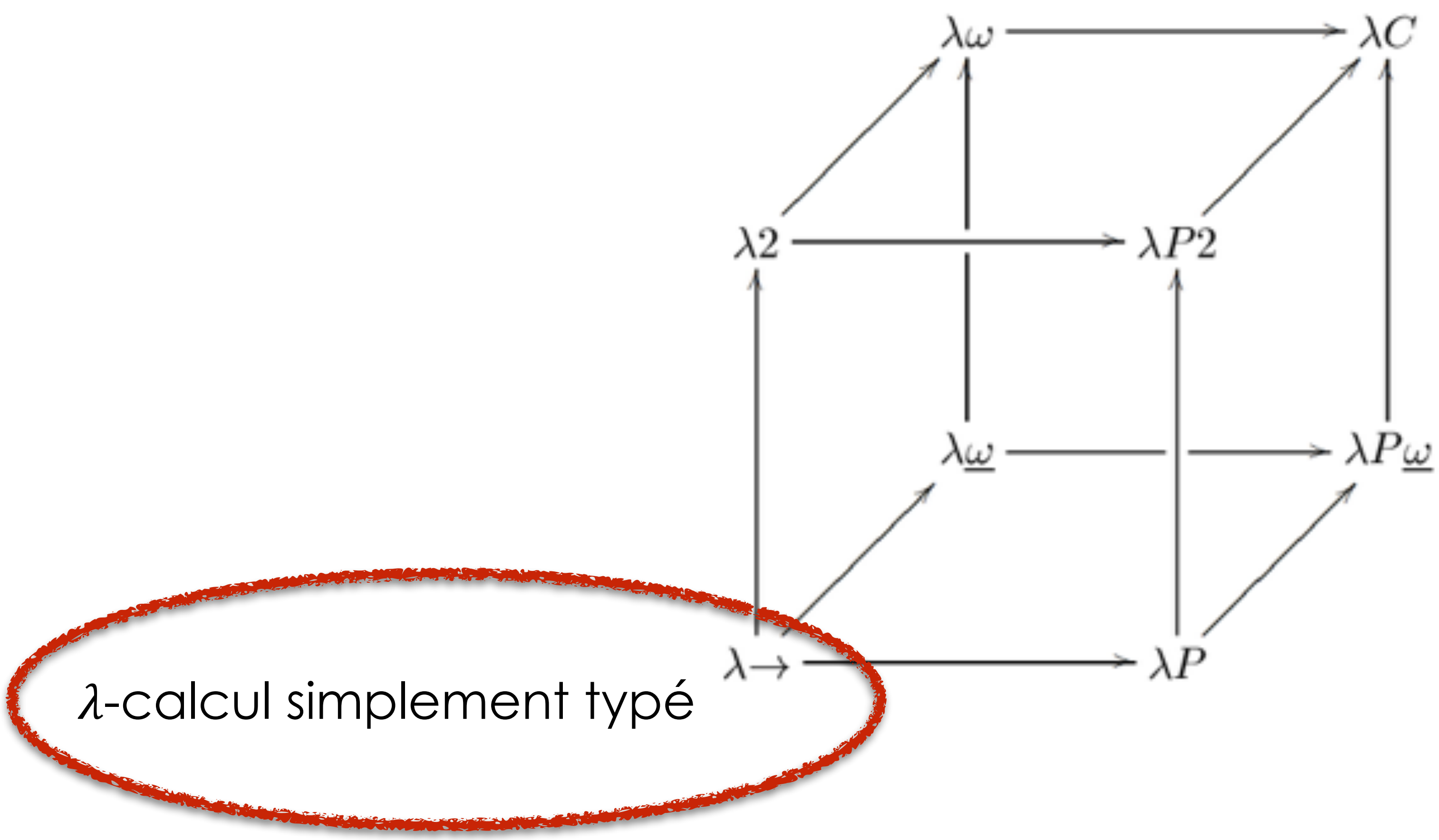
applications : t u

$$\frac{\Gamma, t : \alpha \to \beta \qquad \Gamma \vdash x : \alpha}{\Gamma \vdash tx : \beta}$$

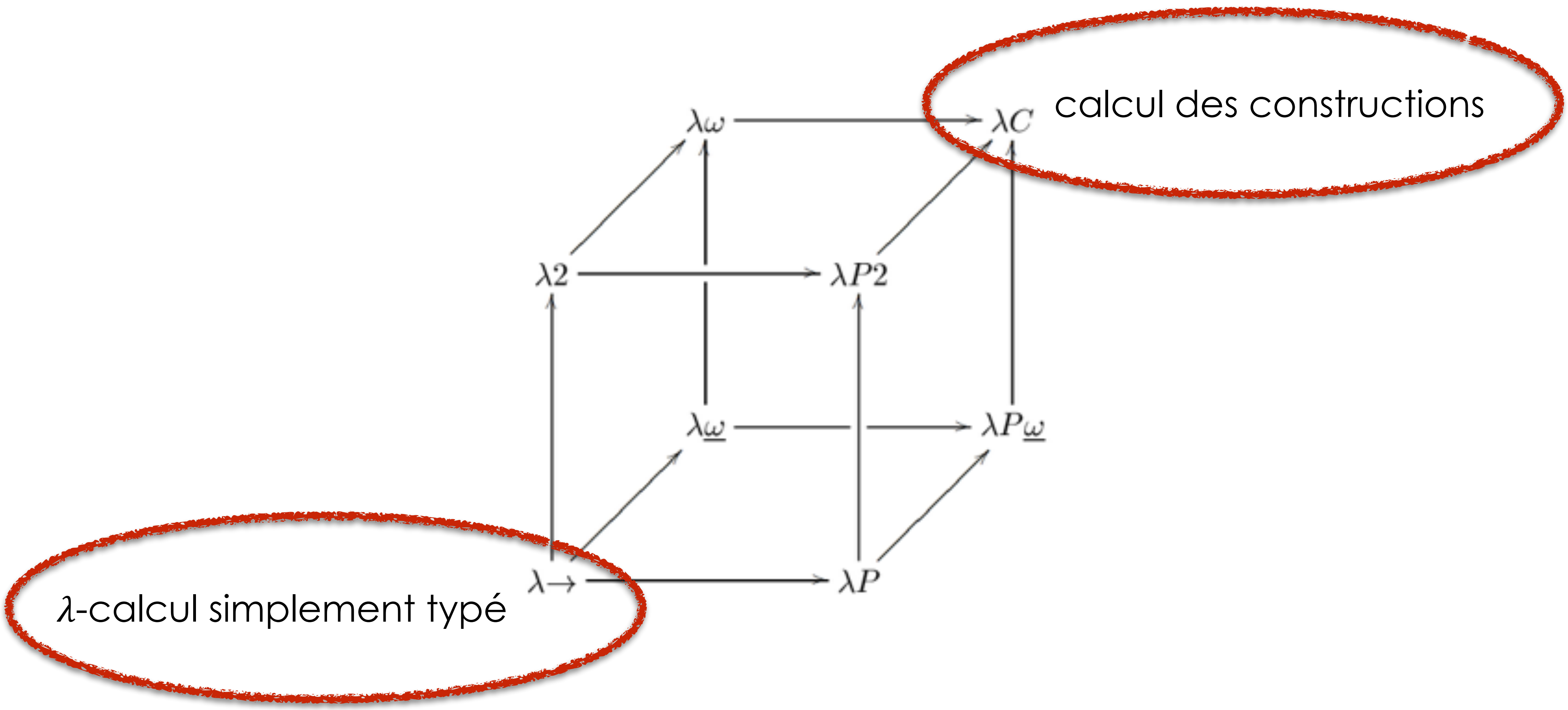
λ-cube de Barendregt



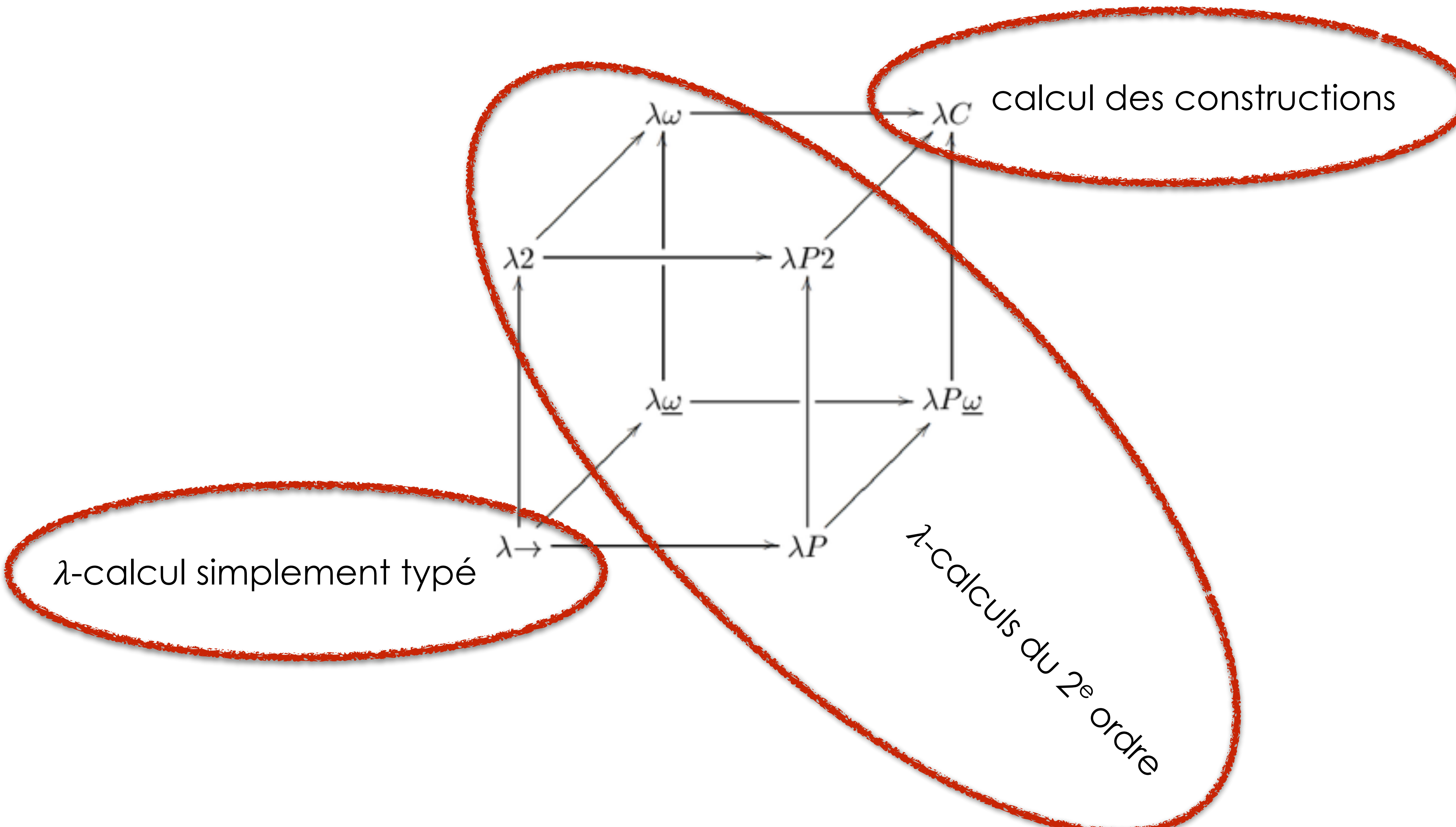
λ-cube de Barendregt



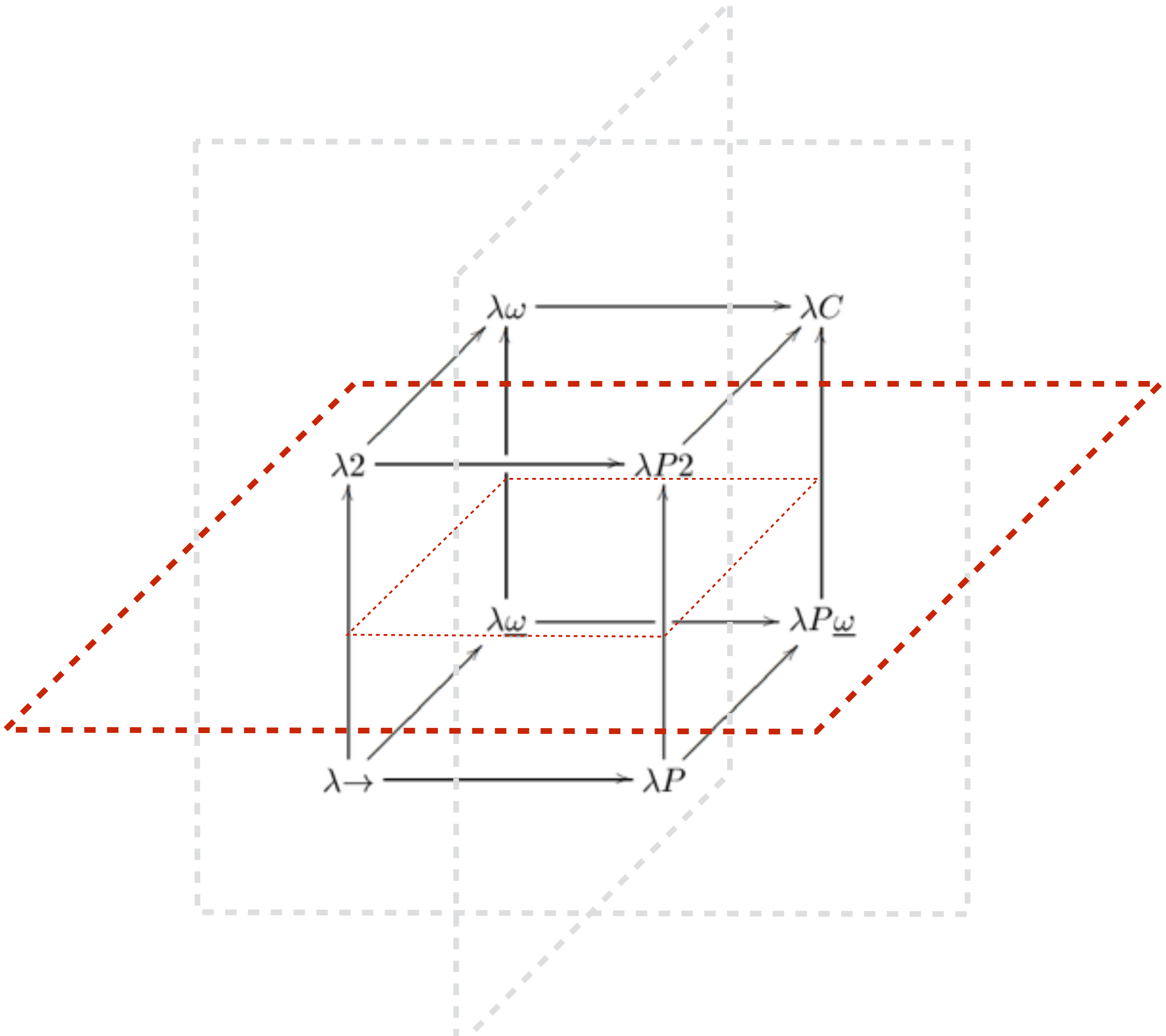
λ-cube de Barendregt



λ-cube de Barendregt



λ-cube de Barendregt

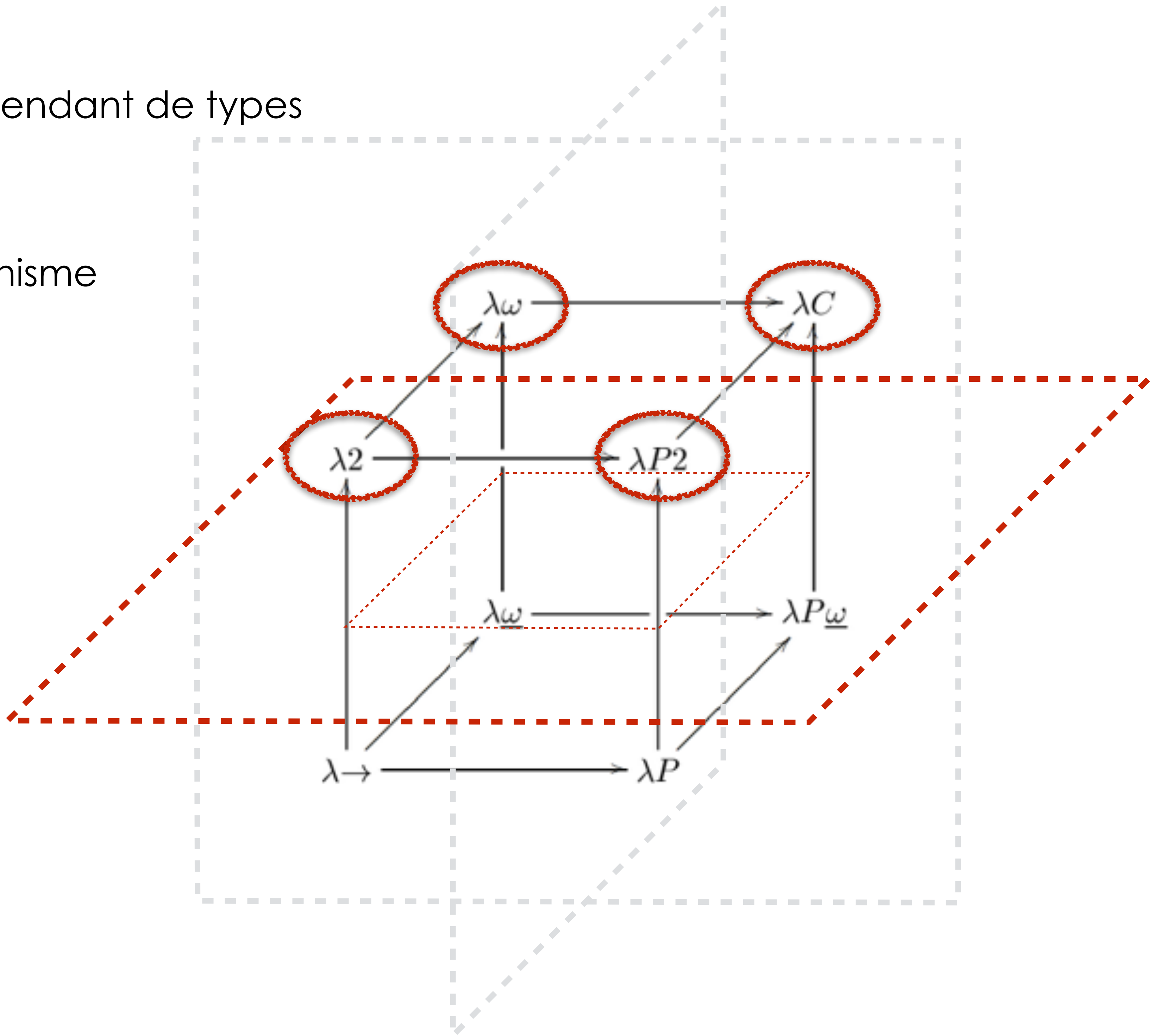


λ-cube de Barendregt

Termes dépendant de types

$\Lambda\alpha.x : \forall\alpha.\alpha$

= Polymorphisme

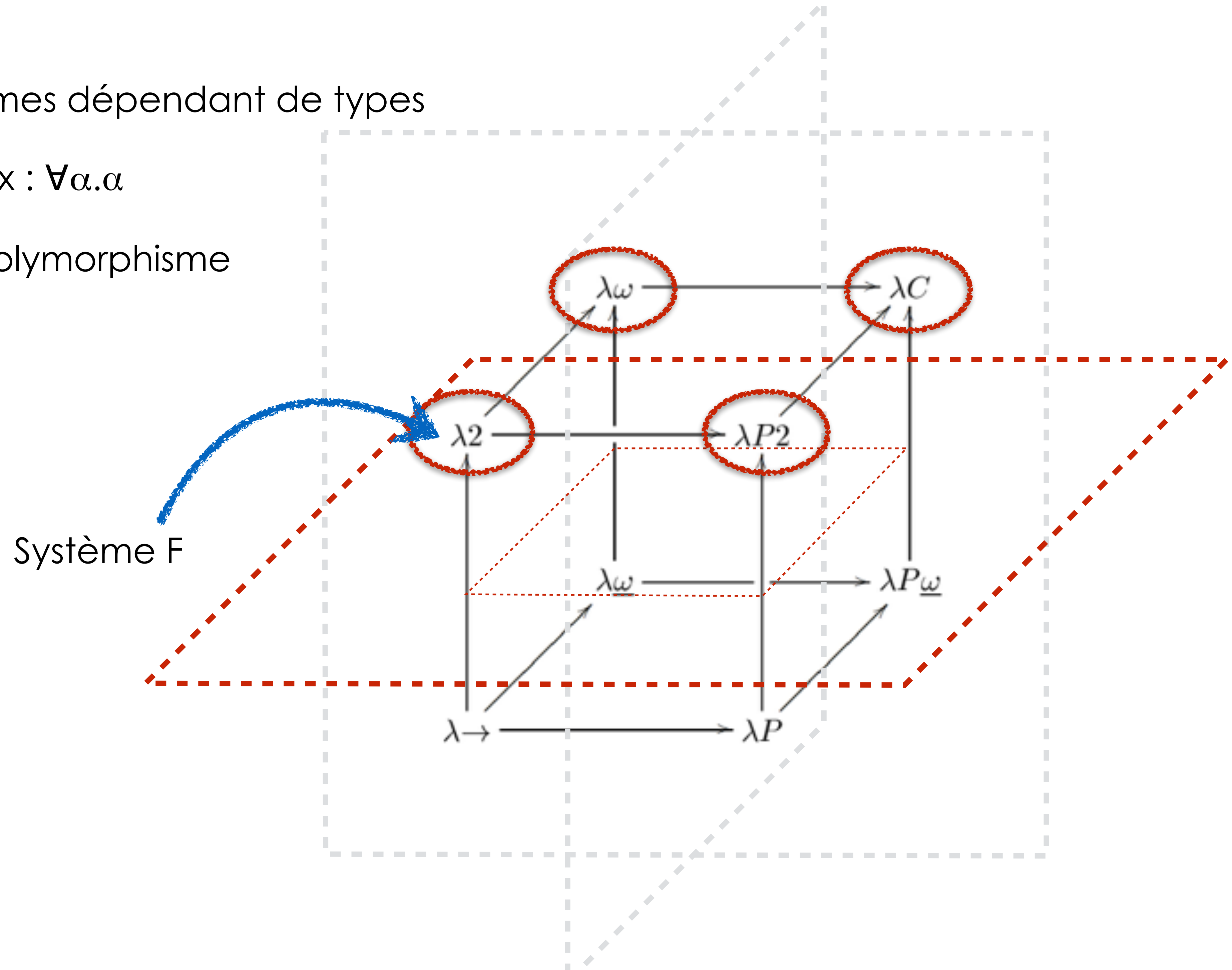


λ-cube de Barendregt

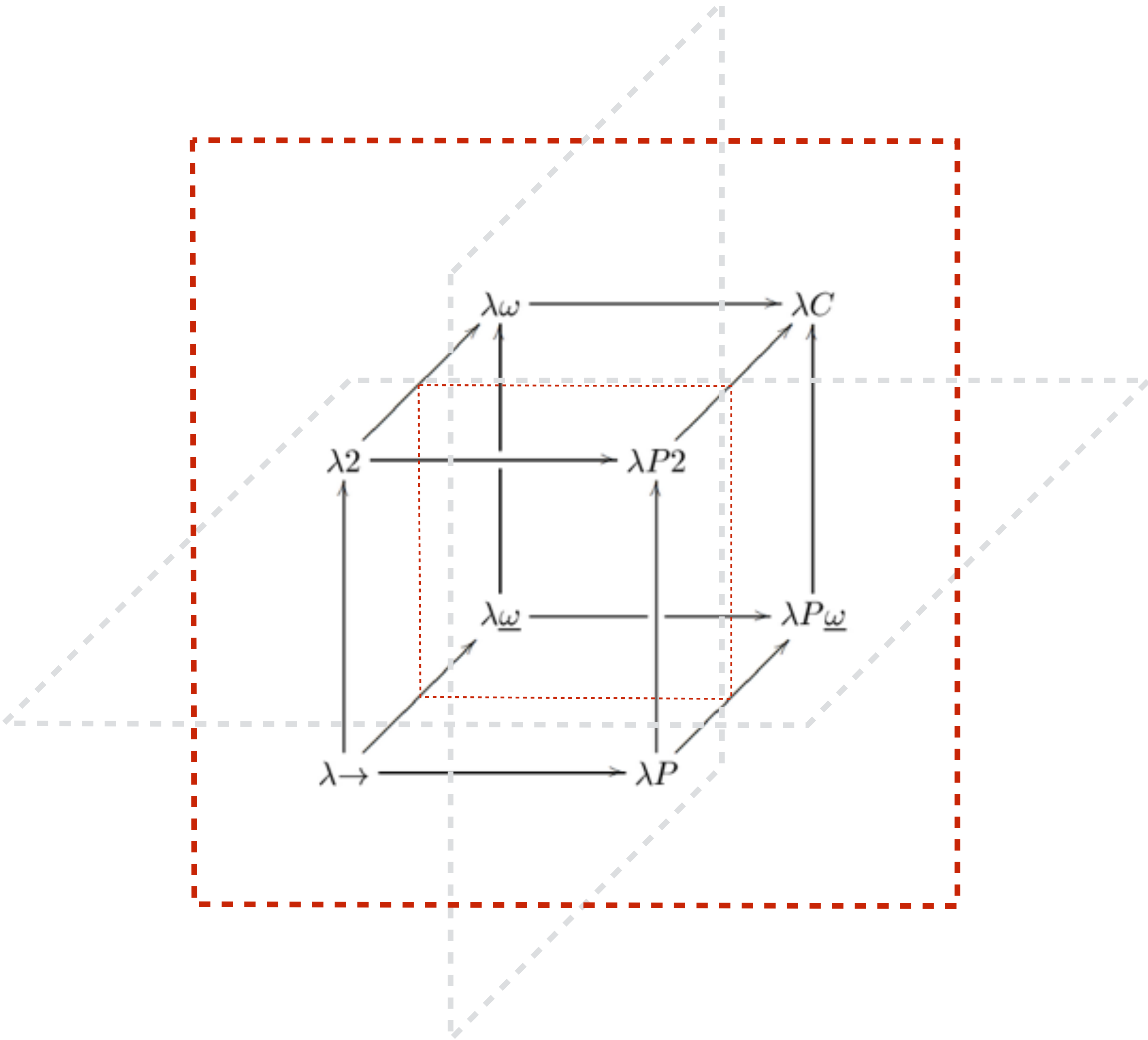
Termes dépendant de types

Λα.x : ∀α.α

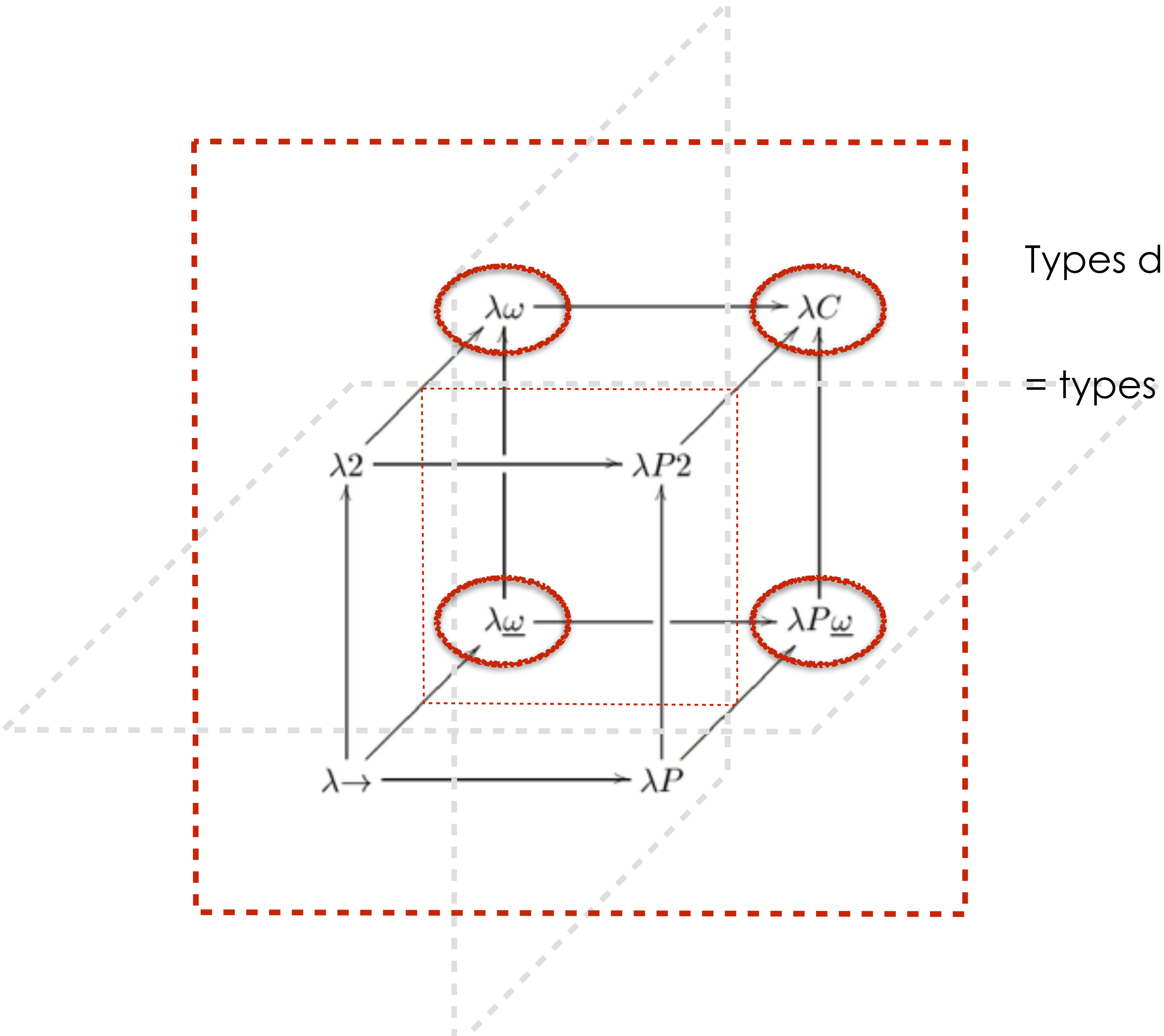
= Polymorphisme



λ-cube de Barendregt



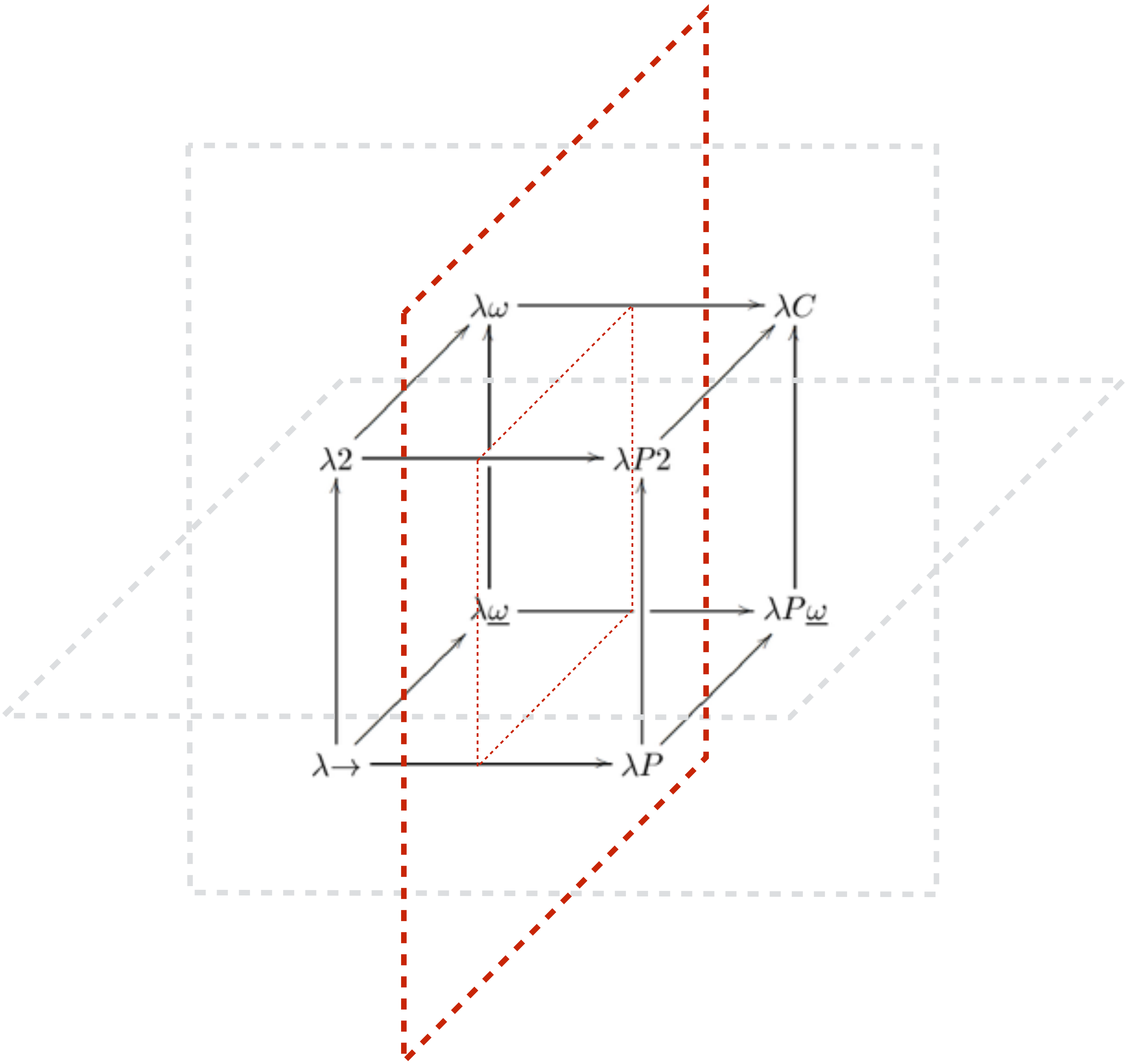
λ-cube de Barendregt



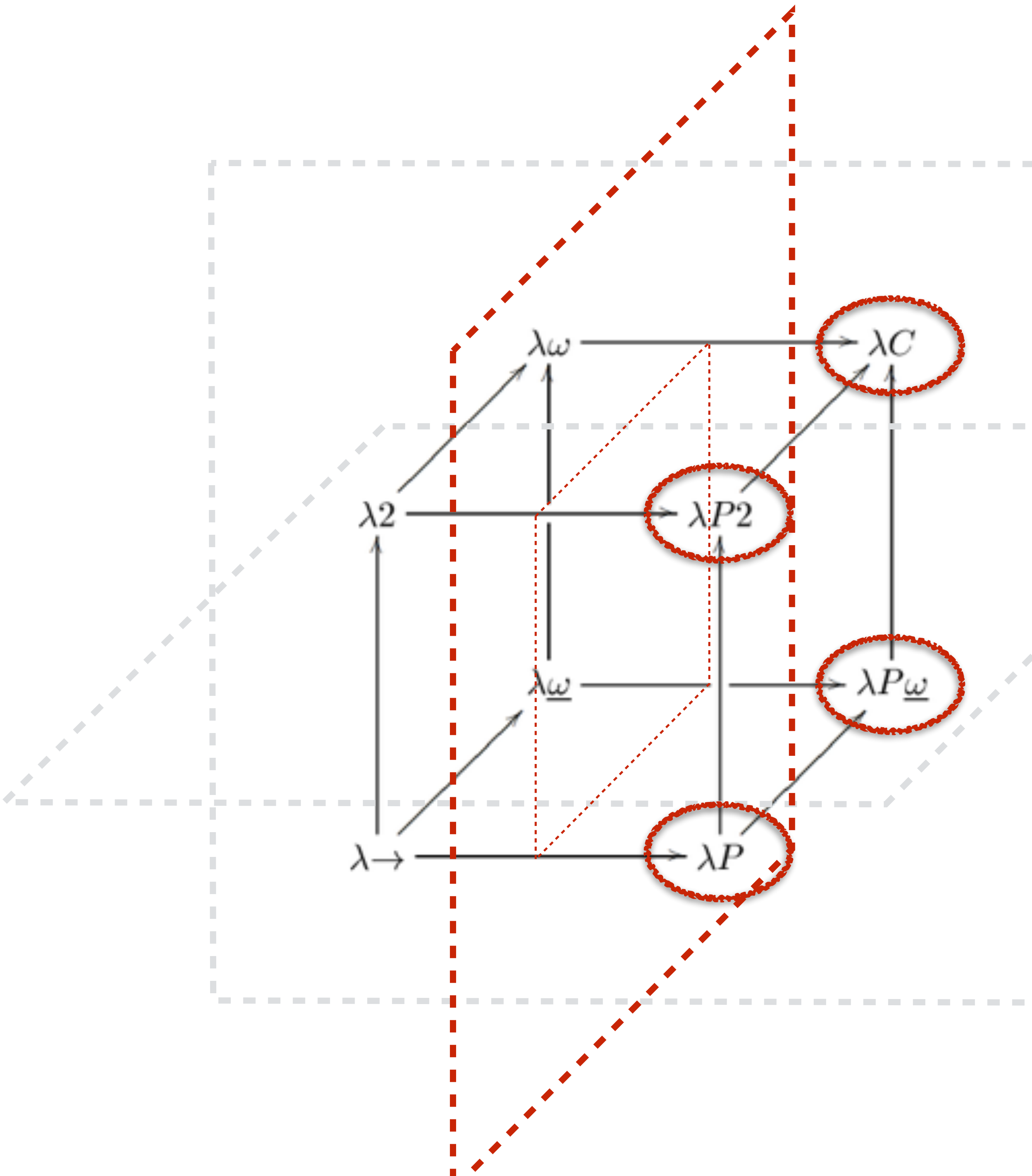
Types dépendant de types

= types récursifs

λ-cube de Barendregt



λ-cube de Barendregt



Types dépendant de termes

x : $(\Pi\alpha{:}\alpha.\beta)$

Formules - Types

Système Logique		**Langage**
$\top$	$\Leftrightarrow$	Type unit
$\bot$	$\Leftrightarrow$	Exceptions
Variable	$\Leftrightarrow$	Type de base
Conjonction	$\Leftrightarrow$	Tuple
Disjonction	$\Leftrightarrow$	Type somme
Implication	$\Leftrightarrow$	Fonction
Pour tout	$\Leftrightarrow$	Type dépendant

Variables - types de base

$$\frac{}{\Gamma, A \vdash A}\ \mathrm{ax}$$

$$\frac{}{\Gamma, x : \alpha \vdash x : \alpha}$$

Implications - Fonctions

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \Rightarrow_i \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B} \Rightarrow_e$$

$$\frac{\Gamma, x : \alpha \vdash t : \beta}{\Gamma \vdash (\lambda x : \alpha . t) : (\alpha \rightarrow \beta)} \rightarrow_i \qquad \frac{\Gamma, t : \alpha \rightarrow \beta \qquad \Gamma \vdash x : \alpha}{\Gamma \vdash tx : \beta} \rightarrow_e$$

Application

Conjonctions - Paires

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \wedge_i \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \wedge_{eg} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B} \wedge_{ed}$$

$$\frac{\Gamma \vdash x_1 : \alpha \qquad \Gamma \vdash x_2 : \beta}{\Gamma \vdash (x_1, x_2) : \alpha \times \beta} \times_i \qquad \frac{\Gamma \vdash (x_1, x_2) : \alpha \times \beta}{\Gamma \vdash x_1 : \alpha} \times_{eg} \qquad \frac{\Gamma \vdash (x_1, x_2) : \alpha \times \beta}{\Gamma \vdash x_2 : \beta} \times_{ed}$$

Projections π_1 et π_2

Disjonctions - Somme

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \vee_{ig} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \vee_{id} \qquad \frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \theta \qquad \Gamma, \psi \vdash \theta}{\Gamma \vdash \theta} \vee_e$$

$$\frac{\Gamma \vdash x : \alpha}{\Gamma \vdash y : \alpha \mid \beta} |_{ig} \qquad \frac{\Gamma \vdash x : \beta}{\Gamma \vdash y : \alpha \mid \beta} |_{id} \qquad \frac{\Gamma \vdash x : \alpha \mid \beta \qquad \Gamma, x : \alpha \vdash z : \gamma \qquad \Gamma, y : \beta \vdash z : \gamma}{\Gamma \vdash z : \gamma} |_e$$

Pattern matching

Pour tout - variables de type

$$\frac{\Gamma \vdash \varphi \qquad x \text{ n'est pas libre dans } \Gamma}{\Gamma \vdash \forall x.\varphi} \forall_i \qquad \frac{\Gamma \vdash \forall x.\varphi}{\Gamma \vdash \varphi[t/x]} \forall_e$$

$$\frac{\Gamma \vdash t : \sigma \qquad \alpha \notin vl(\Gamma)}{\Gamma \vdash \Lambda\alpha.t : (\forall\alpha.\sigma)} \forall_i \qquad \frac{\Gamma \vdash t : (\forall\alpha.\sigma)}{\Gamma \vdash t\tau : (\sigma[\tau/\alpha])} \forall_e$$

Exemple : curryfication

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\alpha \times \beta) \to \gamma}\,ax \qquad \dfrac{\dfrac{}{\Gamma \vdash \alpha}\,ax \qquad \dfrac{}{\Gamma \vdash \beta}\,ax}{\Gamma \vdash \alpha \times \beta}\,\times_i}{\Gamma = (\alpha \times \beta) \to \gamma,\ \alpha,\ \beta \vdash \gamma}\,\to_e}{(\alpha \times \beta) \to \gamma,\ \alpha \vdash \beta \to \gamma}\,\to_i}{((\alpha \times \beta) \to \gamma) \vdash \alpha \to (\beta \to \gamma)}\,\to_i}{\vdash ((\alpha \times \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))}\,\to_i$$

Exemple : curryfication

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f : (\alpha \times \beta) \to \gamma}\,ax \qquad \dfrac{\dfrac{}{\Gamma \vdash a : \alpha}\,ax \qquad \dfrac{}{\Gamma \vdash b : \beta}\,ax}{\Gamma \vdash (a,b) : \alpha \times \beta}\,\times_i}{\Gamma = f : (\alpha \times \beta) \to \gamma, a : \alpha, b : \beta \vdash f(a,b) : \gamma}\,\to_e}{f : (\alpha \times \beta) \to \gamma, a : \alpha \vdash \lambda b.f(a,b) : \beta \to \gamma}\,\to_i}{f : ((\alpha \times \beta) \to \gamma) \vdash \lambda a.\lambda b.f(a,b) : \alpha \to (\beta \to \gamma)}\,\to_i}{\vdash \lambda f.\lambda a.\lambda b.f(a,b) : ((\alpha \times \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))}\,\to_i
$$

Exemple : décurryfication

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma \vdash \alpha \to (\beta \to \gamma)}\,ax
      \qquad
      \dfrac{\dfrac{}{\Gamma \vdash \alpha \times \beta}\,ax}{\Gamma \vdash \alpha}\,\times_{eg}
    }{\Gamma \vdash \beta \to \gamma}\,\to_e
    \qquad
    \dfrac{\dfrac{}{\Gamma \vdash \alpha \times \beta}\,ax}{\Gamma \vdash \beta}\,\times_{ed}
  }{\Gamma = \alpha \to (\beta \to \gamma),\ \alpha \times \beta \vdash \gamma}\,\to_e
}{
  \dfrac{\alpha \to (\beta \to \gamma) \vdash ((\alpha \times \beta) \to \gamma))}{\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \times \beta) \to \gamma))}\,\to_i
}\,\to_i
$$

Exemple : décurryfication

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\Gamma \vdash f : \alpha \to (\beta \to \gamma)}\, ax
        \qquad
        \dfrac{\dfrac{}{\Gamma \vdash c : \alpha \times \beta}\, ax}{\Gamma \vdash \pi_1 c : \alpha}\, \times_{eg}
      }{\Gamma \vdash f(\pi_1 c) : \beta \to \gamma}\, \to_e
      \qquad
      \dfrac{\dfrac{}{\Gamma \vdash c : \alpha \times \beta}\, ax}{\Gamma \vdash \pi_2 c : \beta}\, \times_{ed}
    }{\Gamma = f : \alpha \to (\beta \to \gamma), c : \alpha \times \beta \vdash fc : \gamma}\, \to_e
  }{f : \alpha \to (\beta \to \gamma) \vdash (\lambda c.fc) : ((\alpha \times \beta) \to \gamma))}\, \to_i
}{\vdash (\lambda f.\lambda c.fc) : (\alpha \to (\beta \to \gamma)) \to ((\alpha \times \beta) \to \gamma))}\, \to_i
$$